Problem 16.5

(a) Show that $u = g(x + ct)$ is a solution of the wave equation (16.4) for any twice differentiable function $g(\xi)$. **(b)** Argue clearly that this solution represents a disturbance that travels undistorted to the left.

Solution

Part (a)

The wave equation is given by equation (16.4) on page 684.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{16.4}$$

Find the derivatives of the given function $u(x, t) = g(x + ct)$ by using the chain rule.

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t} g(x + ct) = g'(x + ct)\frac{\partial}{\partial t}(x + ct) = g'(x + ct)(c) = cg'(x + ct)$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial t}\left[cg'(x + ct)\right] = cg''(x + ct)\frac{\partial}{\partial t}(x + ct) = cg''(x + ct)(c) = c^2 g''(x + ct)$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} g(x + ct) = g'(x + ct)\frac{\partial}{\partial x}(x + ct) = g'(x + ct)(1) = g'(x + ct)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x} g'(x + ct) = g''(x + ct)\frac{\partial}{\partial x}(x + ct) = g''(x + ct)(1) = g''(x + ct)$$

Notice that

$$\frac{\partial^2 u}{\partial t^2} = c^2 g''(x + ct) = c^2 \frac{\partial^2 u}{\partial x^2}.$$

Therefore, $u(x, t) = g(x + ct)$ is a solution of the wave equation.

Part (b)

Plugging in $t = 0$ gives the initial waveform.

$$u(x, 0) = g(x)$$

At $t = 1/c$,

$$u\left(x, \frac{1}{c}\right) = g(x + 1),$$

the graph is exactly the same but translated to the left by 1 unit. At $t = 2/c$,

$$u\left(x, \frac{2}{c}\right) = g(x + 2),$$

the graph is exactly the same but translated to the left by 2 units. The more time that passes, the farther to the left the initial wave moves; the wave is said to be moving with speed c.

As an example, consider a travelling Gaussian pulse.

$$u(x,t) = e^{-(x+ct)^2}$$

Below is a plot of $u(x,t)$ versus x for six times.